

EXTENSIONS OF RAMANUJAN'S TWO FORMULAS FOR $1/\pi$ ^ACHUANAN WEI, ^BDIANXUAN GONG^A*Department of Information Technology
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ABSTRACT. In terms of the hypergeometric method, we establish the extensions of two formulas for $1/\pi$ due to Ramanujan [27]. Further, other five summation formulas for $1/\pi$ with free parameters are also derived in the same way.

1. INTRODUCTION

For a complex number x and an integer n , define the shifted factorial by

$$(x)_n = \Gamma(x+n)/\Gamma(x)$$

where Γ -function is well-defined:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \text{ with } \operatorname{Re}(x) > 0.$$

For centuries, the study of π -formulas attracts many mathematicians. The corresponding results can be found in [1]-[3], [5]-[15] and [18]-[30]. Thereinto, two formulas for $1/\pi$ due to Ramanujan [27] can be stated as

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{(k!)^3} \frac{6k+1}{4^k} = \frac{4}{\pi}, \quad (1)$$

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (\frac{1}{4})_k (\frac{3}{4})_k}{(k!)^3} \frac{8k+1}{9^k} = \frac{2\sqrt{3}}{\pi}. \quad (2)$$

Following Bailey [4], define the hypergeometric series by

$${}_1F_s \left[\begin{matrix} a_0, & a_1, & \dots, & a_r \\ b_1, & \dots, & b_s \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \dots (a_r)_k}{k! (b_1)_k \dots (b_s)_k} z^k.$$

Then the identity due to Gessel-Stanton [17, Eq. (1.7)] and Gasper's identity (cf. [16, Eq. (5.23)]) can be expressed as

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{3}, b, 1 - b, c, \frac{1}{2} + a - c + n, -n \\ \frac{a}{3}, \frac{2+a-b}{2}, \frac{1+a+b}{2}, 1 + a - 2c, 1 + a + 2n, 2c - a - 2n \end{matrix} \middle| 1 \right] \\ &= \frac{(\frac{1+a}{2})_n (1 + \frac{a}{2})_n (\frac{1+a+b}{2} - c)_n (1 + \frac{a-b}{2} - c)_n}{(\frac{1+a+b}{2})_n (1 + \frac{a-b}{2})_n (\frac{1+a}{2} - c)_n (1 + \frac{a}{2} - c)_n}, \end{aligned} \quad (3)$$

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} 3a, 1 + \frac{3a}{4}, \frac{1-3b}{2}, \frac{2-3b}{2}, 3b, 2a + b + n, -n \\ \frac{3a}{4}, \frac{1+3a+3b}{2}, \frac{3a+3b}{2}, 1 + a - b, 1 - 3a - 3b - 3n, 1 + 3a + 3n \end{matrix} \middle| 1 \right] \\ &= \frac{(a + 2b)_n (a + \frac{1}{3})_n (a + \frac{2}{3})_n (a + 1)_n}{(1 + a - b)_n (a + b)_n (a + b + \frac{1}{3})_n (a + b + \frac{2}{3})_n}. \end{aligned} \quad (4)$$

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The main aim of the paper is to explore the relations of hypergeometric series and π -formulas. Four summation formulas for $1/\pi$ with free parameters including the extension of (1) will be derived from (3) in section 2. Three summation formulas for $1/\pi$ with free parameters including the extension of (2) will be deduced from (4) in section 3.

2. SUMMATION FORMULAS FOR $1/\pi$ WITH FREE PARAMETERS IMPLIED BY THE GESSEL-STANTON IDENTITY

Letting $n \rightarrow \infty$ for (3), we obtain the following equation:

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} a, 1 + \frac{a}{3}, b, 1 - b, c \\ \frac{a}{3}, \frac{2+a-b}{2}, \frac{1+a+b}{2}, 1 + a - 2c \end{matrix} \middle| \frac{1}{4} \right] \\ &= \frac{\Gamma(\frac{1+a+b}{2})\Gamma(1 + \frac{a-b}{2})\Gamma(\frac{1+a}{2} - c)\Gamma(1 + \frac{a}{2} - c)}{\Gamma(\frac{1+a}{2})\Gamma(1 + \frac{a}{2})\Gamma(\frac{1+a+b}{2} - c)\Gamma(1 + \frac{a-b}{2} - c)}. \end{aligned} \quad (5)$$

Choosing $a = \frac{1}{2} + 2p$, $b = \frac{1}{2} + 2q$ and $c = \frac{1}{2} + r$ in (5), we achieve the extension of (1).

Theorem 1. *For $p, q, r \in \mathbb{Z}$ with $\min\{p+q, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:*

$$\begin{aligned} \frac{1}{\pi} &= \frac{(\frac{1}{2})_{p+q-r}(\frac{1}{2})_{p-q-r}}{(\frac{1}{2})_r} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{k+2p}(\frac{1}{2})_{k+2q}(\frac{1}{2})_{k-2q}(\frac{1}{2})_{k+r}}{k!(k+p+q)!(k+p-q)!(\frac{1}{2})_{k+2p-2r}} \frac{6k+4p+1}{4^{k+r+1}}. \end{aligned}$$

When $p = q = r = 0$, Theorem 1 reduces to Ramanujan's formula for $1/\pi$ given by (1) exactly. Other two examples of the same type are displayed as follows.

Example 1 ($p = q = 1, r = 2$ in Theorem 1).

$$\frac{256}{3\pi} = \sum_{k=0}^{\infty} \frac{(\frac{5}{2})_k^3}{(k!)^2(k+2)!} \frac{6k+5}{4^k}.$$

Example 2 ($p = q = 2, r = 4$ in Theorem 1).

$$\frac{16384}{315\pi} = \sum_{k=0}^{\infty} \frac{(\frac{9}{2})_k^3}{(k!)^2(k+4)!} \frac{2k+3}{4^k}.$$

Making $a = \frac{3}{2} + 2p$, $b = \frac{3}{2} + 2q$ and $c = \frac{1}{2} + r$ in (5), we attain the identity.

Theorem 2. *For $p, q, r \in \mathbb{Z}$ with $\min\{p+q+1, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:*

$$\begin{aligned} \frac{1}{\pi} &= \frac{(\frac{1}{2})_{p+q-r+1}(\frac{1}{2})_{p-q-r}}{(\frac{1}{2})_r} \\ &\times \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_{k+2p}(\frac{3}{2})_{k+2q}(-\frac{1}{2})_{k-2q}(\frac{1}{2})_{k+r}}{k!(k+p+q+1)!(k+p-q)!(\frac{3}{2})_{k+2p-2r}} \frac{6k+4p+3}{4^{k+r+1}}. \end{aligned}$$

Two examples from Theorem 2 are laid out as follows.

Example 3 ($p = q = 0, r = 1$ in Theorem 2).

$$\frac{32}{3\pi} = \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_k^3}{(k!)^2(k+1)!} \frac{2k+1}{4^k}.$$

Example 4 ($p = q = 1, r = 3$ in Theorem 2).

$$\frac{2048}{15\pi} = \sum_{k=0}^{\infty} \frac{(\frac{7}{2})_k^3}{(k!)^2(k+3)!} \frac{6k+7}{4^k}.$$

Taking $a = \frac{1}{2} + 2p$, $b = \frac{1}{2} + 2q$ in (5) and then letting $c \rightarrow -\infty$, we get the identity.

Theorem 3. For $p, q \in \mathbb{Z}$ with $\min\{p+q, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:

$$\frac{4^{p+1}}{\sqrt{2}\pi} = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{k+2p}(\frac{1}{2})_{k+2q}(\frac{1}{2})_{k-2q}}{k!(k+p+q)!(k+p-q)!} \frac{6k+4p+1}{(-8)^k}.$$

Two examples from Theorem 3 are displayed as follows.

Example 5 ($p = q = 0$ in Theorem 3).

$$\frac{2\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{(k!)^3} \frac{6k+1}{(-8)^k}.$$

Example 6 ($p = q = 1$ in Theorem 3).

$$\frac{32\sqrt{2}}{3\pi} = \sum_{k=0}^{\infty} \frac{(\frac{5}{2})_k^2(-\frac{3}{2})_k}{(k!)^2(k+2)!} \frac{6k+5}{(-8)^k}.$$

Setting $a = \frac{3}{2} + 2p$, $b = \frac{3}{2} + 2q$ in (5) and then letting $c \rightarrow -\infty$, we gain the identity.

Theorem 4. For $p, q \in \mathbb{Z}$ with $\min\{p+q+1, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:

$$\frac{4^{p+2}}{\sqrt{2}\pi} = \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_{k+2p}(\frac{3}{2})_{k+2q}(-\frac{1}{2})_{k-2q}}{k!(k+p+q+1)!(k+p-q)!} \frac{6k+4p+3}{(-8)^k}.$$

Two examples from Theorem 4 are laid out as follows.

Example 7 ($p = q = 0$ in Theorem 4).

$$\frac{8\sqrt{2}}{3\pi} = \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_k^2(-\frac{1}{2})_k}{(k!)^2(k+1)!} \frac{2k+1}{(-8)^k}.$$

Example 8 ($p = q = 1$ in Theorem 4).

$$\frac{128\sqrt{2}}{15\pi} = \sum_{k=0}^{\infty} \frac{(\frac{7}{2})_k^2(-\frac{5}{2})_k}{(k!)^2(k+3)!} \frac{6k+7}{(-8)^k}.$$

3. SUMMATION FORMULAS FOR $1/\pi$ WITH FREE PARAMETERS IMPLIED BY GASPER'S IDENTITY

Letting $n \rightarrow \infty$ for (4), we obtain the following equation:

$$\begin{aligned} {}_5F_4 \left[\begin{matrix} 3a, 1 + \frac{3a}{4}, 3b, \frac{1-3b}{2}, \frac{2-3b}{2} \\ \frac{3a}{4}, 1 + a - b, \frac{1+3a+3b}{2}, \frac{3a+3b}{2} \end{matrix} \middle| \frac{1}{9} \right] \\ = \frac{\Gamma(1+a-b)\Gamma(a+b)\Gamma(a+b+\frac{1}{3})\Gamma(a+b+\frac{2}{3})}{\Gamma(a+2b)\Gamma(a+\frac{1}{3})\Gamma(a+\frac{2}{3})\Gamma(a+1)}. \end{aligned} \quad (6)$$

Choosing $a = \frac{1}{6} + p$ and $b = \frac{1}{6} + q$ in (6), we achieve the extension of (2).

Theorem 5. For $p, q \in \mathbb{Z}$ with $\min\{p+q, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:

$$\frac{2(-1)^q}{3^{3q-\frac{1}{2}}\pi} = \left(\frac{1}{2}\right)_{p+2q} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_{k+3p}(\frac{1}{2})_{k+3q}(\frac{1}{2})_{2k-3q}}{k!(k+p-q)!(2k+3p+3q)!} \frac{8k+6p+1}{9^k}.$$

When $p = q = 0$, Theorem 5 reduces to Ramanujan's formula for $1/\pi$ offered by (2) exactly. Other two examples of the same type are displayed as follows.

Example 9 ($p = q = 1$ in Theorem 5).

$$\frac{1024\sqrt{3}}{405\pi} = \sum_{k=0}^{\infty} \frac{(\frac{7}{2})_k(-\frac{5}{4})_k(-\frac{3}{4})_k}{(k!)^2(k+3)!} \frac{8k+7}{9^k}.$$

Example 10 ($p = q = 2$ in Theorem 7).

$$\frac{524288\sqrt{3}}{7577955\pi} = \sum_{k=0}^{\infty} \frac{(\frac{13}{2})_k(-\frac{11}{4})_k(-\frac{9}{4})_k}{(k!)^2(k+6)!} \frac{8k+13}{9^k}.$$

Taking $a = \frac{1}{2} + p$ and $b = \frac{1}{2} + q$ in (6), we attain the identity.

Theorem 6. For $p, q \in \mathbb{Z}$ with $\min\{p+q, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:

$$\frac{4(-1)^q}{3^{3q+\frac{1}{2}}\pi} = \left(\frac{1}{2}\right)_{p+2q+1} \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_{k+3p}(\frac{3}{2})_{k+3q}(-\frac{1}{2})_{2k-3q}}{k!(k+p-q)!(2k+3p+3q+2)!} \frac{8k+6p+3}{9^k}.$$

Two examples from Theorem 6 are laid out as follows.

Example 11 ($p = q = 0$ in Theorem 6).

$$\frac{16\sqrt{3}}{3\pi} = \sum_{k=0}^{\infty} \frac{(\frac{3}{2})_k(-\frac{1}{4})_k(\frac{1}{4})_k}{(k!)^2(k+1)!} \frac{8k+3}{9^k}.$$

Example 12 ($p = q = 1$ in Theorem 6).

$$\frac{8192\sqrt{3}}{8505\pi} = \sum_{k=0}^{\infty} \frac{(\frac{9}{2})_k(-\frac{7}{4})_k(-\frac{5}{4})_k}{(k!)^2(k+4)!} \frac{8k+9}{9^k}.$$

Setting $a = \frac{5}{6} + p$ and $b = \frac{5}{6} + q$ in (6), we get the identity.

Theorem 7. For $p, q \in \mathbb{Z}$ with $\min\{p+q+1, p-q\} \geq 0$, there holds the summation formula for $1/\pi$ with free parameters:

$$\frac{8(-1)^q}{3^{3q+\frac{5}{2}}\pi} = \left(\frac{1}{2}\right)_{p+2q+2} \sum_{k=0}^{\infty} \frac{(\frac{5}{2})_{k+3p}(\frac{5}{2})_{k+3q}(-\frac{3}{2})_{2k-3q}}{k!(k+p-q)!(2k+3p+3q+4)!} \frac{8k+6p+5}{9^k}.$$

Two examples from Theorem 7 are displayed as follows.

Example 13 ($p = q = 0$ in Theorem 7).

$$\frac{128\sqrt{3}}{27\pi} = \sum_{k=0}^{\infty} \frac{(\frac{5}{2})_k(-\frac{3}{4})_k(-\frac{1}{4})_k}{(k!)^2(k+2)!} \frac{8k+5}{9^k}.$$

Example 14 ($p = q = 1$ in Theorem 7).

$$\frac{65536\sqrt{3}}{229635\pi} = \sum_{k=0}^{\infty} \frac{(\frac{11}{2})_k(-\frac{9}{4})_k(-\frac{7}{4})_k}{(k!)^2(k+5)!} \frac{8k+11}{9^k}.$$

Remark: With the change of the parameters, Theorems 1-7 can produce more concrete formulas for $1/\pi$. We shall not lay them out here.

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